THE STABILITY OF MOTION OF A GYROSCOPE

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In the papers [1-4]* the authors consider the motion of a symmetric gyroscope with its center of gravity on the axis of spin, and derive by various methods the sufficient conditions of stability of motion in the two cases: (1) when the axis of the outer ring is vertical [1-3] (In the paper [3] the author establishes the necessary condition as well); (2) when the axis of the outer ring is horizontal [4].

In this note the author derives from Chetaev's theorem on the instability of motion the necessary condition of stability in case (2).

Let X_1, Y_1, Z_1, be the fixed coordinate system (the Z_1-axis being vertical), and let x, y, z be the moving coordinate system (the z-axis coinciding with the spin axis).

Let J be the moment of inertia of the outer ring about the fixed axis Z_1, A', B', C' be moments of inertia of the inner ring about the axes x, y, z respectively, ζ be the distance between the center of gravity of the gyroscope and the origin of the moving coordinate system, θ be the angle of nutation of the gyroscope, ψ be the angle of rotation of the outer ring about the axis Z_1, ϕ be the angle of revolution of the rotor about the spin axis z, and $\dot\theta$, $\dot\psi$, $\dot\phi$ be the co responding angular celocities.

In the case (2) the kinetic energy of the system is

$$T = \frac{1}{2} J\dot\psi^2 + \frac{1}{2}(A'\dot\theta^2 + B'\dot\psi^2 \sin^2\theta + C'\dot\psi^2\cos^2\theta) +$$
$$+ \frac{1}{2}[A\dot\theta^2 + A\dot\psi^2 \sin^2\theta + C(\dot\varphi + \dot\psi\cos\theta)^2]$$

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* See also: Skimel, V.N., Nekotorye zadachi ob ustoichivosti dvizheniia tverdego tela (Certain problems of stability of motion of a rigid body). *Avtoreferat dissertatsii.* Kazan, 1955.

The force function of the system is $U = -mg\,\zeta \sin\theta \sin\psi$. Let the generalized coordinates of the gyroscope be q_1, q_2, q_3, where

$$\psi = q_1, \quad \theta = q_2, \quad \varphi = q_3$$

and the generalized momenta be p_1, p_2, p_3; then the Hamiltonian of the system can be written as

$$H = \frac{1}{2}\frac{(p_1 - p_3\cos\theta)^2}{(J + B'\sin^2\theta + C'\cos^2\theta + A\sin^2\theta)} + \frac{1}{2}\frac{p_2^2}{(A + A')} + \frac{1}{2}\frac{p_3^2}{C} + mg\zeta\sin\theta\sin\psi$$

The canonical equations of motion of the gyroscope

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial p_i} \qquad (i = 1, 2, 3)$$

have the first integrals

$$p_3 = \text{const}, \qquad H = \text{const}$$

The unperturbed motion of the gyroscope corresponds to the following particular solution of the canonical equations:

$$q_{10} = \frac{1}{2}\pi, \quad q_{20} = \frac{1}{2}\pi, \quad q_{30} = q_{30}t; \qquad p_{10} = 0, \quad p_{20} = 0, \quad p_{30} = cr_0$$

where $q_3 + q_1\cos\theta = r_0$ is a constant. In order to construct the equations of the perturbed motion we shall introduce the following:

$$q_i = q_{i0} + \xi_i, \qquad p_i = p_{i0} + \eta_i \qquad (i = 1, 2, 3)$$

where ξ, η, are the perturbations.

The equations of the perturbed motion have the integral $H - H_0$ = constant, where H and H_0 are the Hamiltonians with

$$q_i = q_{i0} + \xi_i, \quad p_i = p_{i0} + \eta_i \quad (i = 1, 2, 3), \qquad q_i = q_{i0}, \quad p_i = p_{i0} \quad (i = 1, 2, 3)$$

Expanding $(H - H_0)$ in Taylor series we obtain

$$H - H_0 = \frac{1}{2(J + B' + A)}(\eta_1 + p_{30}\xi_2)^2 + \frac{1}{2}\frac{\eta_2^2}{(A + A')} - \frac{1}{2}mg\zeta(\xi_1^2 + \xi_2^2) + \ldots$$

With the abbreviations

$$\frac{1}{J + B' + A} = a, \qquad \frac{1}{A + A'} = b, \qquad mg\zeta = e, \qquad p_{30} = p$$

we can write the equations of motion in the Poincaré variations as

$$\frac{d\xi_1}{dt} = a(\eta_1 + p\xi_2), \qquad \frac{d\xi_2}{dt} = b\eta_2, \qquad \frac{d\xi_3}{dt} = 0$$

$$\frac{d\eta_1}{dt} = e\xi_1, \qquad \frac{d\eta_2}{dt} = (e - ap^2)\,\xi_2 - ap\eta_1, \qquad \frac{d\eta_3}{dt} = 0$$

According to the above equations the integral $H - H_0$ can be rewritten as

$$H - H_0 = \frac{1}{2a}\dot{\xi}_1^2 + \frac{1}{2b}\dot{\xi}_2^2 - \frac{e}{2}\xi_1^2 - \frac{e}{2}\xi_2^2 + \dots$$

which would not be determinable when $\zeta > 0$. In such a case the Poincaré degree of instability would equal two. Hence, on the strength of Kelvin's theorem, the gyroscopic stability is possible.

The first integral of the equations of motion in Poincaré variations was given by Chetaev [5] as

$$\Gamma = 2\left(\frac{e}{a}\dot{\xi}_1\xi_2 - \frac{e}{b}\xi_1\dot{\xi}_2\right) - pe(\xi_1^2 + \xi_2^2) +$$
$$+ \frac{e(a-b)}{2abp}\left(\frac{1}{a}\dot{\xi}_1^2 - \frac{1}{b}\dot{\xi}_2^2 + e\xi_2^2 - e\xi_1^2\right) = \text{const}$$

The Liapunov function $p(H - H_0) - \Gamma$ will be positive-definite if the following condition is satisfied:

$$(abp^2 - eb - ea)^2 - 4e^2ab > 0$$

or

$$c^2r_0^2 > mg\zeta\,[(J + 2A + A' + B) \pm 2\sqrt{(J + B' + A)(A + A')}]$$

If the integral Γ = const could be extended to the equations of the perturbed motion, then the above inequality would be the sufficient condition for stability [4].

We shall demonstrate now that this condition is a necessary one. Let us consider the function

$$V = e\xi_1\xi_2 - b\eta_1\eta_2$$

Taking into account the equations of the perturbed motion, the derivative of this function is

$$\frac{dV}{dt} = e\dot{\xi}_1\xi_2 + e\xi_1\dot{\xi}_2 - b\dot{\eta}_1\eta_2 - b\eta_1\dot{\eta}_2 = aep\xi_2^2 + (abp^2 - be + ae)\,\xi_2\eta_1 + abp\eta_1^2 + \dots$$

It can be easily seen that in the case when the inequality

$$4e^2ab - (abp^2 - eb - ea)^2 > 0$$

is satisfied, dV/dt is positive-definite, and V could assume positive values. Hence, by Chetaev's theorem on the instability of motion, the unperturbed motion is unstable, which proves the necessity of the above condition.

V.N. Skimel (see footnote) proves the necessity of this condition by analyzing the roots of the characteristic equation corresponding to the system of equations of the perturbed motion.

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